Chaos and Emittance growth due to nonlinear interactions in circular accelerators

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Emittance growth

- External incoherent diffusion, radiation, Intrabeam etc.
- Coherent motion, instability
- Nonlinear diffusion
- Nonlinearlity coupled to external diffusion (noise)

1. Incoherent electron cloud
2. Beam-beam limit
3. Space charge limit

Diffusive or not

- Which system does not have emittance growth?

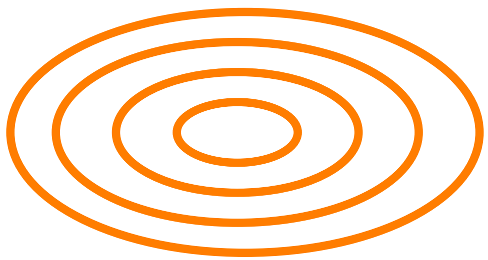
1. Integrable system
2. System with two degrees of freedom

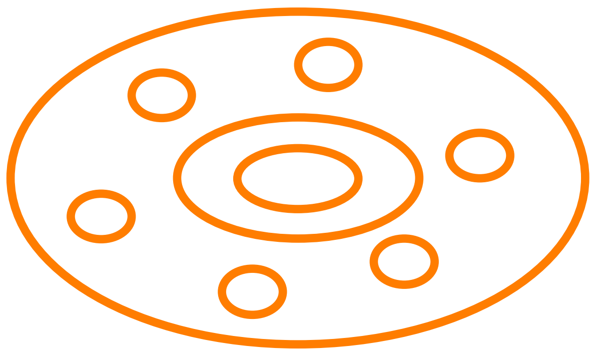
- Which system can have emittance growth?

1. Nonintegrable system with three or more degrees of freedom
2. External diffusion: noise, radiation excitation… The external diffusion is amplified due to nonlinear interaction

Nondiffusive system

- Integrable system – needless to say
- System with two degrees of freedom. Particles do not across torus layers.





System fall into global stochastic regime may be diffusive even for two degrees of freedom, but the diffusion is limited in the regime.

Diffusion in three or more degrees of freedom

- Motion in a degree of freedom gives modulation
- Particles can get over KAM boundary through additional freedom.

Round beam:

Example of non-diffusive system

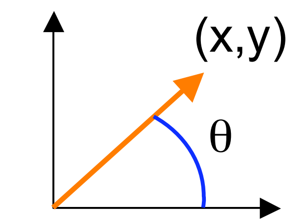
- Equal tune, no synchrotron oscillation -- equivalent to a system with 2 degrees of freedom (r-s).

$$H(x,y,s) = \left[\frac{p_x^2+p_y^2}{2} + \frac{1}{2\beta}\left(x^2+y^2\right)\right] + \delta_P(s)\,U(x,y)$$

$$= \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + \frac{r^2}{2\beta} + \delta_P(s)\,U(r,z)$$

$$p_r = r' = p_x \cos\theta + p_y \sin\theta$$

$$p_\theta = r^2\theta' = r\left(-p_x \sin\theta + p_y \cos\theta\right)$$



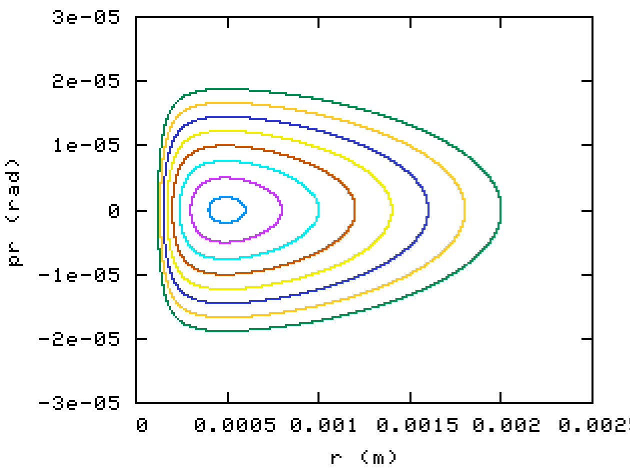
$$z = s - ct$$

- H does not include θ, therefore p_θ is a constant of motion.
- Trajectories on a torus r-p_r-s
- Poincare cross-section is mapped on two dimensional space.

- Motion in r-pr space.

$$J_r = \frac{1}{2\pi}\oint p_r\, dr = \frac{1}{2\pi\beta\mu}\int_{r_{\min}}^{r_{\max}} \sqrt{\frac{E}{} - \frac{r^2 E}{2} - \frac{\ell^2}{r^2}}\, dr \qquad J_\theta = \frac{1}{2\pi}\oint p_\theta\, d\theta = \ell$$

$$\psi_r = \cos^{-1}\frac{r^2 - \beta(2J_r + \ell)}{\beta\sqrt{(2J_r + \ell)^2 - \ell^2}} \qquad \psi_\theta = \theta$$



Model- round beam interacting with electron cloud

- Transverse beam size depends on z; $\sigma_x(z)$, $\sigma_y(z)$.
- Assume transverse Gaussian charge distribution. Applicable not only beam-beam but also electron cloud and space charge issues.
- Strong localized force x, y ~σ_x, σ_y

$$U(x,y,z;\sigma_x,\sigma_y) = -\frac{N r_e}{\gamma}\int_0^\infty \frac{\exp\left(-\frac{x^2}{2\sigma_x^2+u}-\frac{y^2}{2\sigma_y^2+u}\right)}{\sqrt{2\sigma_x^2+u}\sqrt{2\sigma_y^2+u}}\,du$$

$$\mathbf{x}(s) = \exp(-:U:)\,\mathbf{x}(0)$$

$$\mathbf{x} = (x, p_x, y, p_y, z, \delta)^t$$

The force for a round charge distribution

- The force depends on z, because of $\sigma_r(z)$ of strong beam or electron cloud.

$$U(r;\sigma_r(z)) = -\frac{r_e}{\gamma}\int_0^\infty \frac{\exp\left(-\frac{r^2}{2\sigma_r^2+u}\right)}{2\sigma_r^2+u}du$$

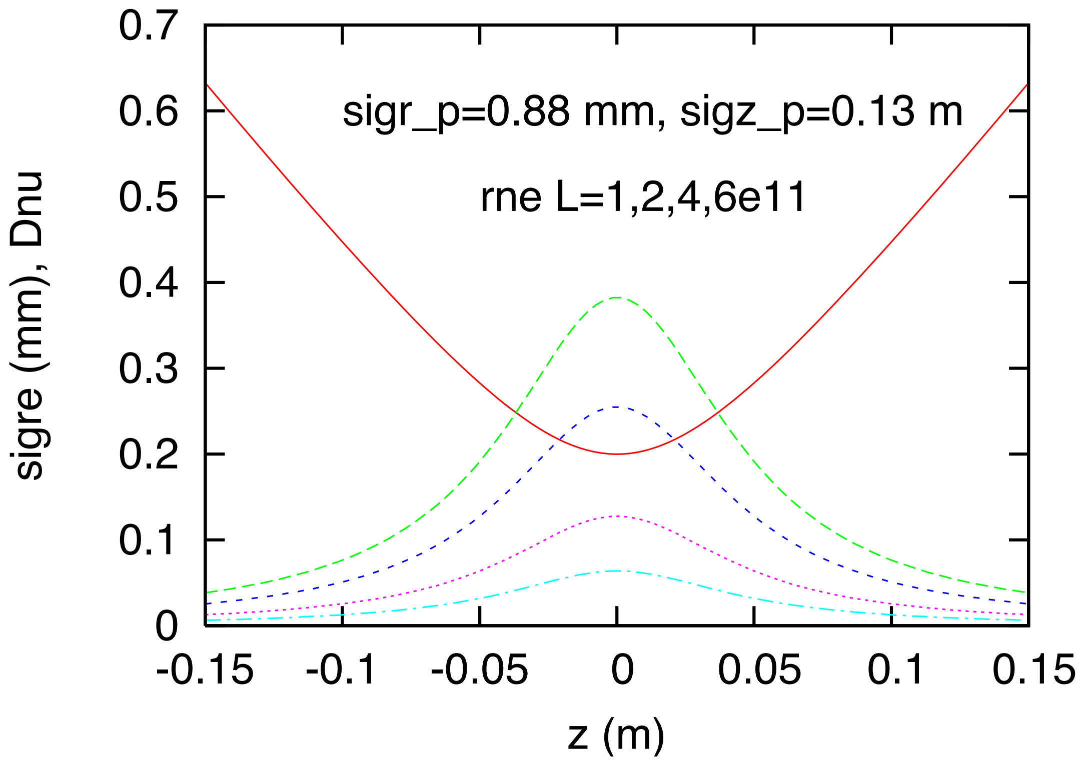
$$F_r = -\frac{\partial U(r;\sigma_r(z))}{\partial r} = -\frac{2r_e}{\gamma}\frac{1}{r}\left[1-\exp\left(-\frac{r^2}{2\sigma_r^2}\right)\right]$$

$$F_z = -\frac{\partial U(r;\sigma_r(z))}{\partial z} = -\frac{\partial U(r;\sigma_r(z))}{\partial \sigma_r^2}\frac{d\sigma_r^2}{dz} = -\frac{2r_e}{\gamma}\frac{1}{2\sigma_r^2}\exp\left(-\frac{r^2}{2\sigma_r^2}\right)\frac{d\sigma_r^2}{dz}$$

- Example, LHC
- L=26700 m, ν_x=0.28, ν_y=0.31, ν_s=0.006
- $\varepsilon_x=\varepsilon_y=8\times10^{-9}$ m, β_x= β_x=100 m
- $\sigma_x=\sigma_y$=0.89 mm, σ_z=0.13 m,

Model of cloud

- σ_r (cloud) and tune shift
- N_e L=1, 2, 4, 6 x10^{11}. Interact at a point in the ring.

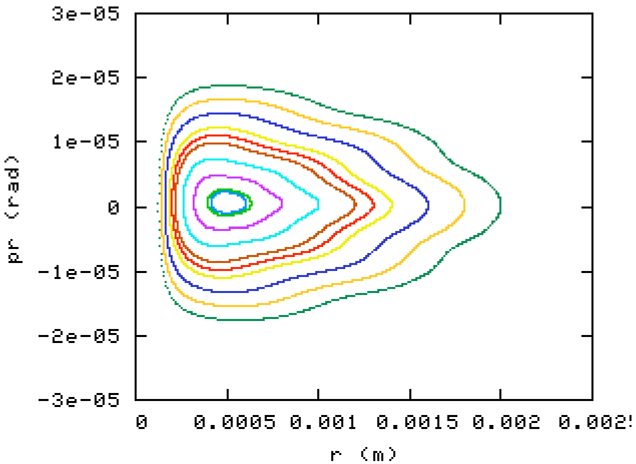


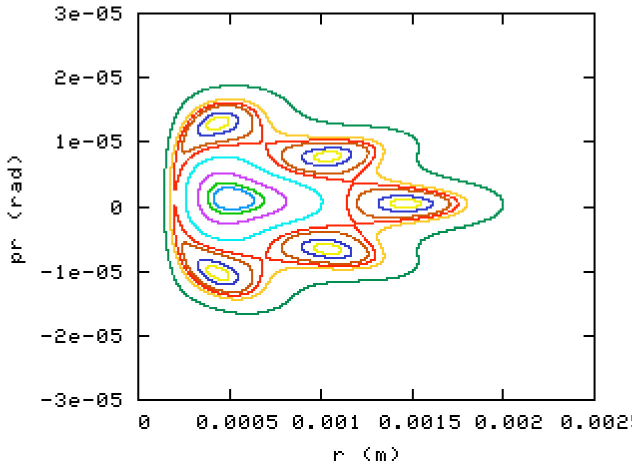
Two degree of freedom

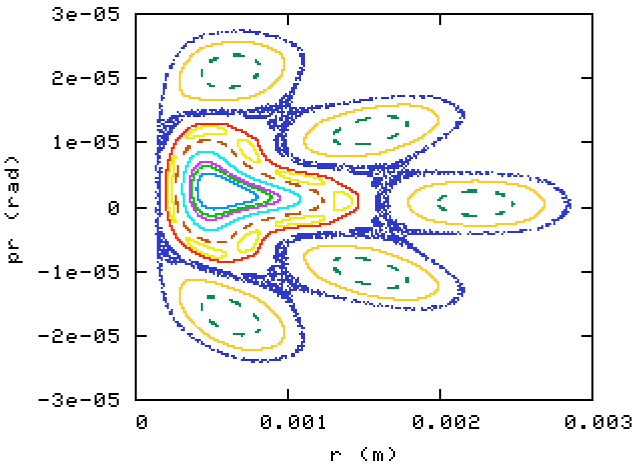
- 4 variable (x, p_x, s, H'). 1 integral for H', Poincare cross section (certain s)→ 2 variable.
- Poincare plot, x-p_x plot at a certain s.
- When one more integral, $J(x,p_x)$=constant, the system is solvable. This relation gives a curve in x-p_x phase space.

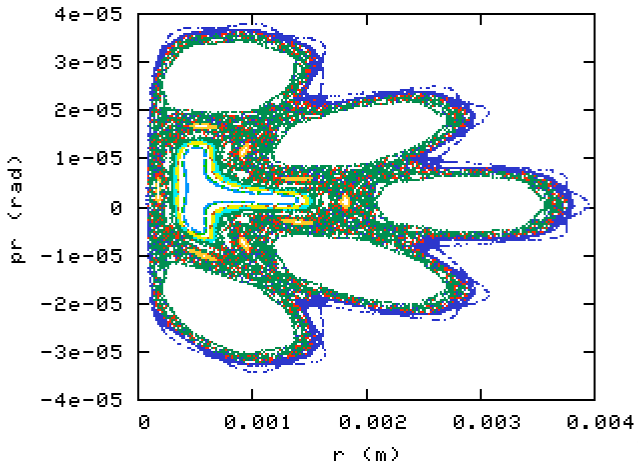
Poincare plot

- r-pr $\Delta\nu$=0.06, 0.13, 0.27, 0.38





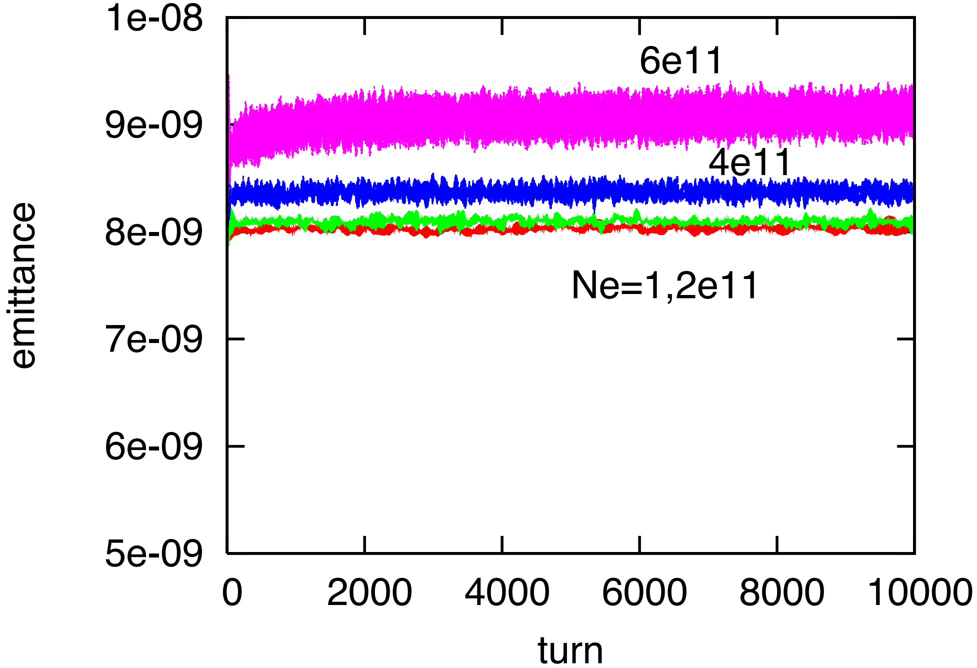




9 mm

Diffusion rate

- Diffusion rates are very small compare than those for 3 degrees of freedom, see later.
- T_0=89 μs.



Resonance overlap

- Fourier expansion for ψ_r of U.

$$U = \sum_{k=0}^{\infty} U_{kr} \cos k\psi$$

- Resonance position, $J_{r,R}$.

$$\mu_r(J_r) = \omega_r + \frac{\partial U_0}{\partial J_r} \qquad \mu_r(J_{r,R})/2\pi = \frac{n}{k}$$

- Motion near the resonance position, pendulum motion, separatrix.

$$I = \frac{J_r - J_{r,R}}{k} \qquad \varphi = m\psi$$

$$H = \frac{1}{2} k^2 \left. \frac{\partial^2 U_0}{\partial J_r^2} \right|_{J_{r,R}} I^2 + U_{rPk} \cos(\varphi)$$

Overlap condition, Chirikov criterion

- Resonance width

$$\Delta J_r = 4\sqrt{U_k \left(\frac{\partial^2 U_0}{\partial J_r^2}\right)^{-1}}$$

- Resonance separation

$$J_{r,k}(k,n+1) - J_{r,k}(k,n) = \frac{2\pi}{k}\left(\frac{\partial^2 U_0}{\partial J_r^2}\right)^{-1}$$

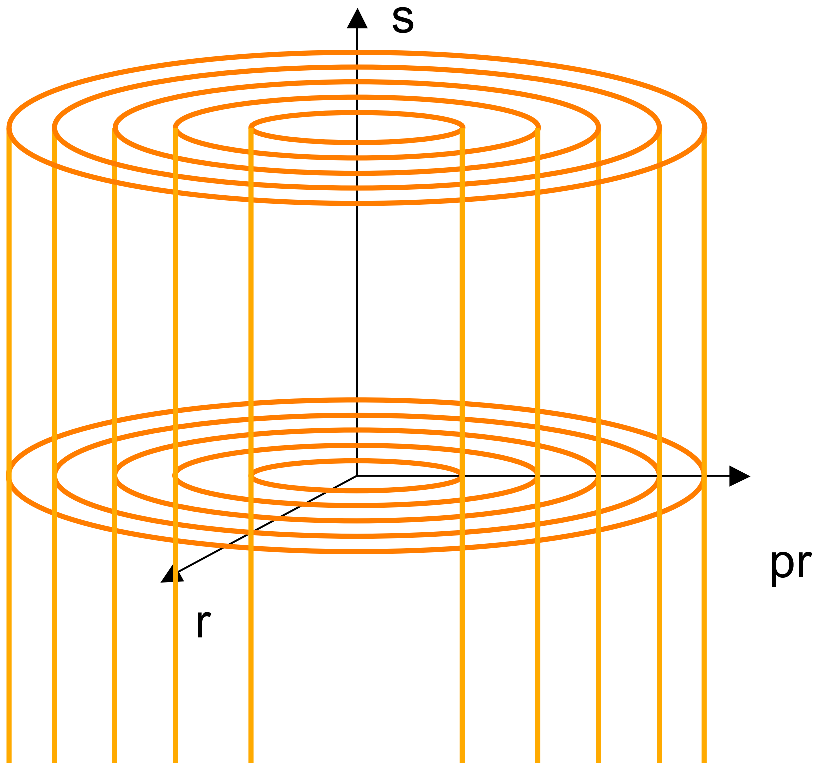
- Overlap condition, width>separation

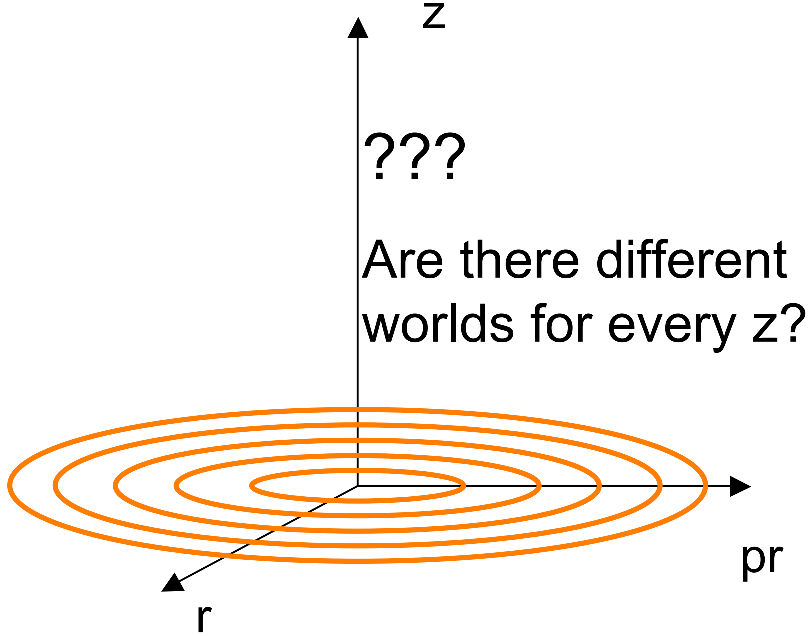
$$k\sqrt{U_k \frac{\partial^2 U_0}{\partial J_r^2}} > \frac{\pi}{2}$$

Synchrotron oscillation and symplectic diffusion

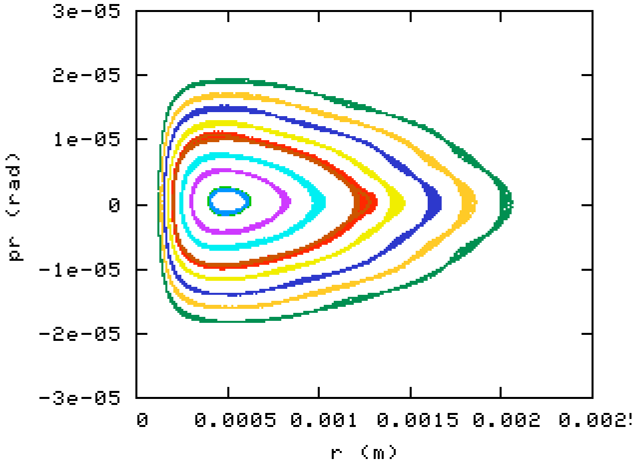
$$H p p z s U \beta \mu \mu \alpha \mu \frac{}{222} \left(\right)^{222}_{rzP} \frac{p_\theta^2}{r^2} \quad \frac{r^2}{\beta\alpha} - \left(\right) \left(\right) -^{2} \qquad ()(,)$$

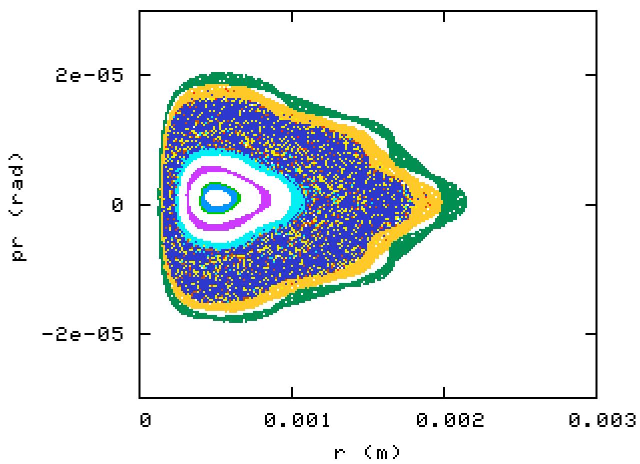
- Add synchrotron motion, 3 degrees of freedom
- The structures of tori are different for each z.
- Are particles return the same torus after one synchrotron period.



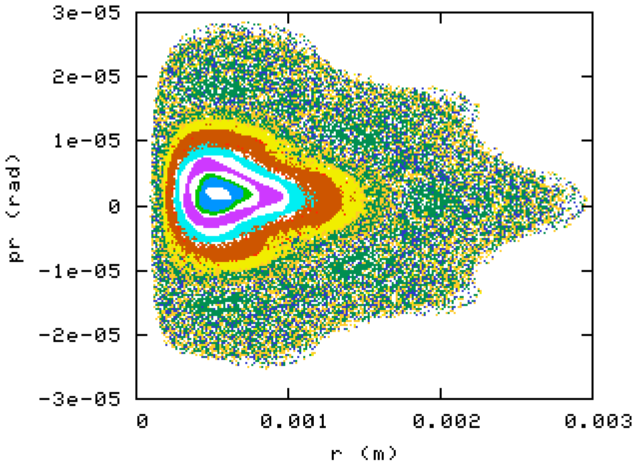


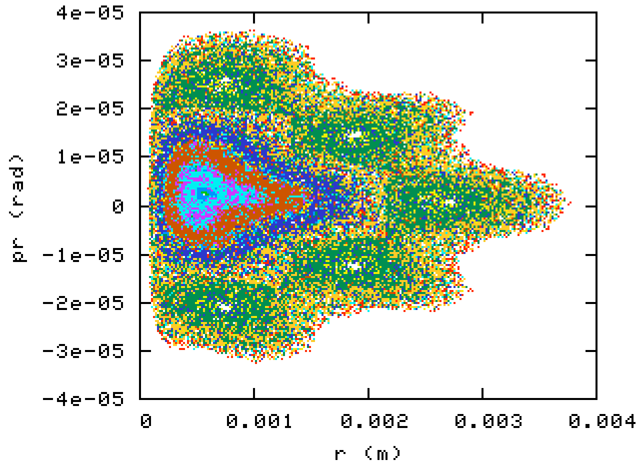
Poincare plot for several z





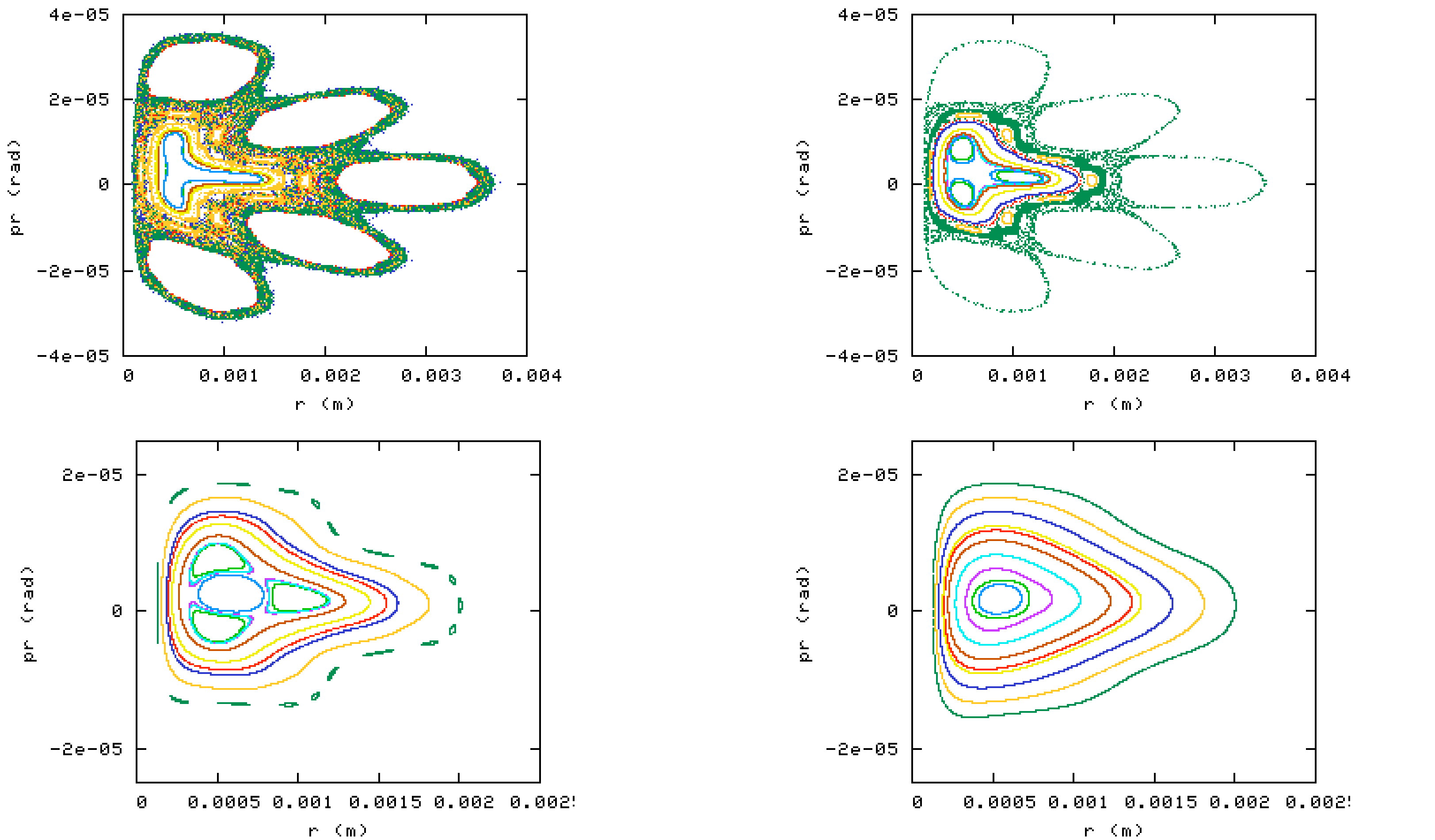
Note σ_r=0.89 mm



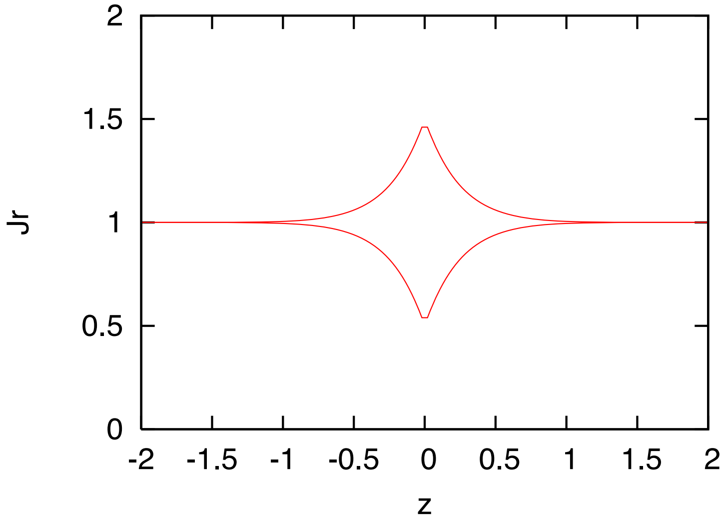


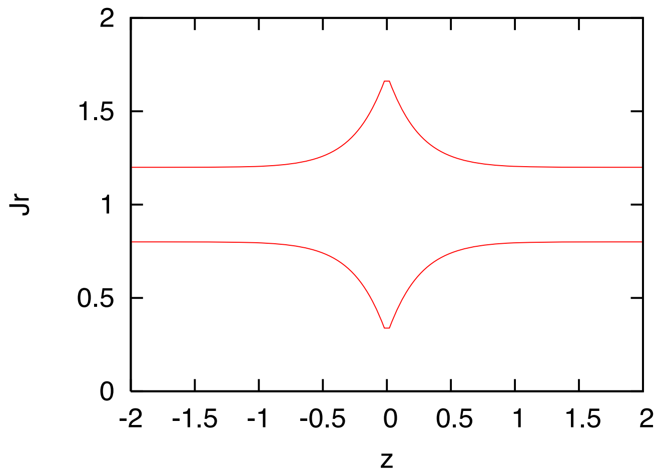
Poincare plot for several z

- Z=2, 4, 6, 10 cm



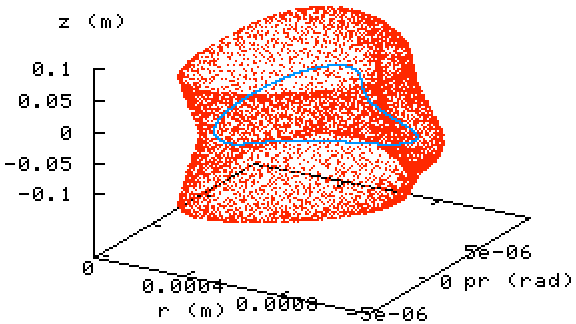
Adiabatic invariant?



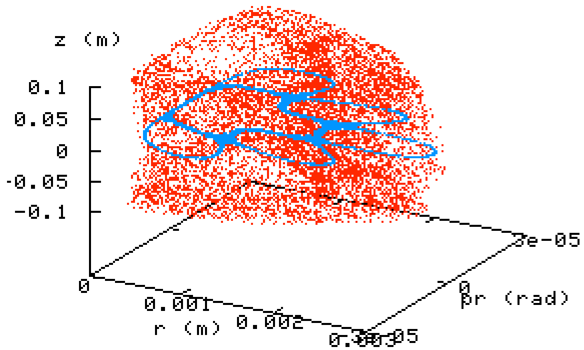


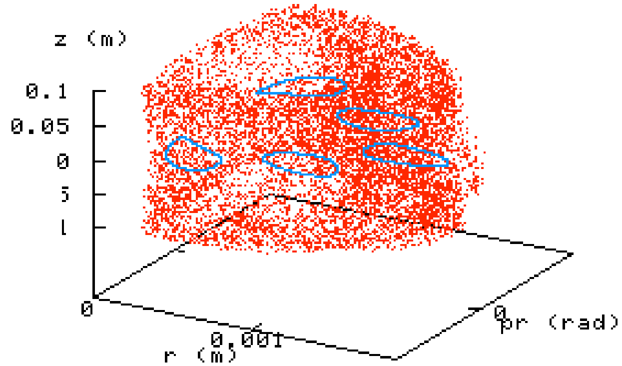
Solvable or not

- 6 variables (x,p,y,p,s,H’)
- One integra H’, Poincare cross section at s.
- 4 variables
- When 2 integrals exist, the system is solvable. The solution is represented by a surface in 3 dimensional space, r-pr-z.
- When a surface is not seen, the system is nonsolvable: i.e., emittance growth occurs.



Solvable





Nonsolvable

Blue: no synchrotron motion at z=0

Separatrix crossing

- U depends on z. Fourier expansion for synchrotron phase.

$$U(J_r, J_z, \psi_r, \psi_z) = U_0(J_r, J_z) + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} U_{kl}(J_r, J_z)\cos\left(k\psi_z - l\psi_r\right)$$

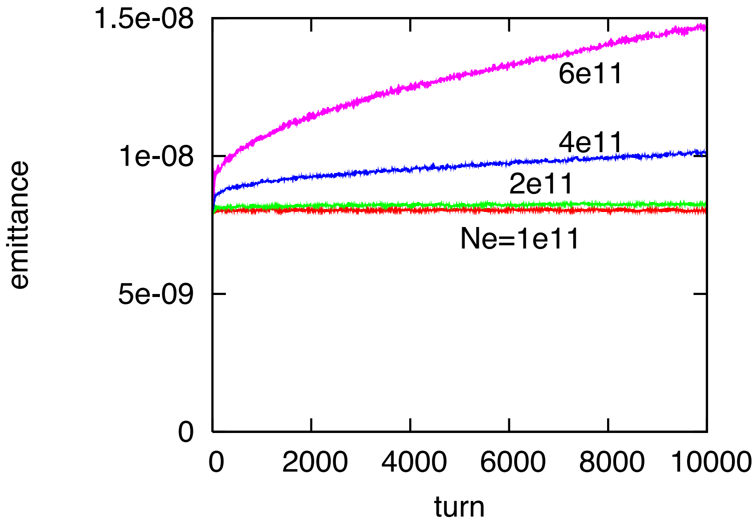
- Resonance separation (narrower than n-n+1)

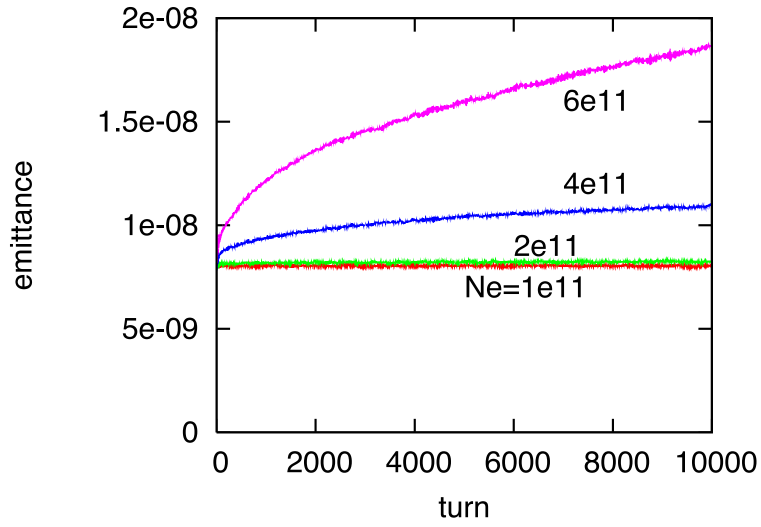
$$\left|J_{r,R}(l+1) - J_{r,R}(l)\right| = \left|\frac{\mu_z}{kJ}\left(\frac{\partial^2 U_0}{\partial J_r^2}\right)^{-1}\right|$$

$$\frac{4k}{\mu_{zr}}\sqrt{U_{kl}\frac{\partial^2 U_0}{\partial J^2}} > 1$$

Diffusion due to synchrotron motion

- ν_z =0.006 =0.012





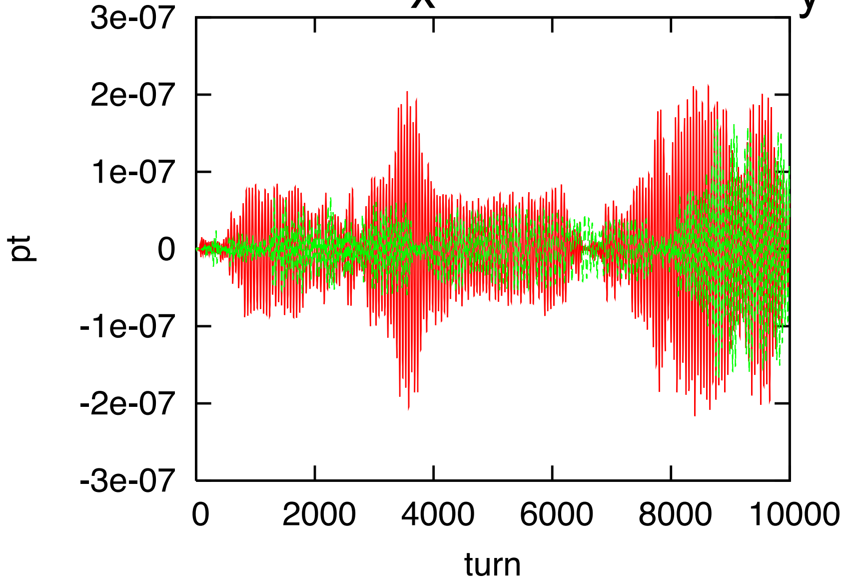
Tune difference

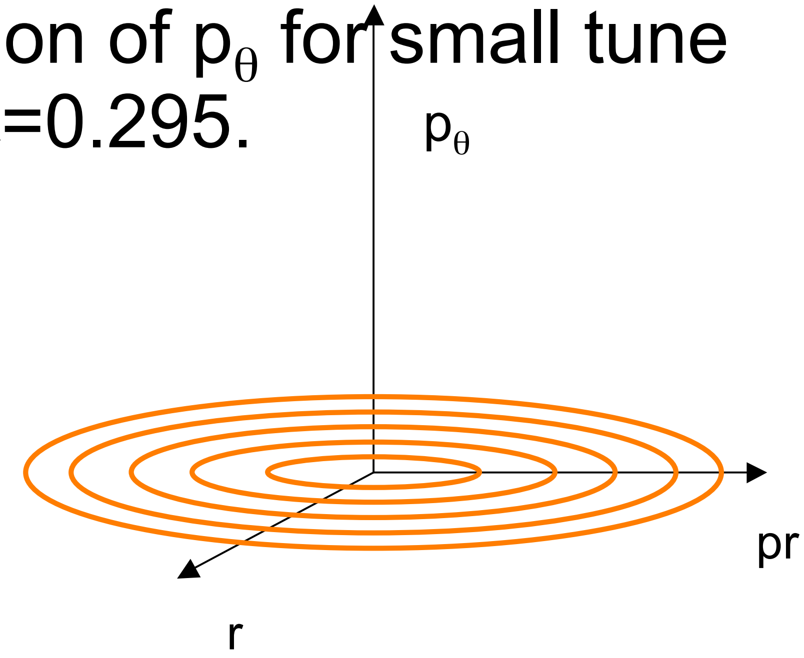
- 3 degrees of freedom

$$H = \mu_x \frac{\beta p_x^2 + x^2/\beta}{2} + \mu_y \frac{\beta p_y^2 + y^2/\beta}{2} + U(x,y;s)$$

$$= \frac{\mu_x+\mu_y}{2}\left(\beta p_r^2 + \beta\frac{p_\theta^2}{r^2} + \frac{r^2}{\beta}\right) + \frac{\delta\mu}{2}\left(\beta p_r^2 - \frac{r^2}{\beta}\right)\cos 2\theta + U(r,z)$$

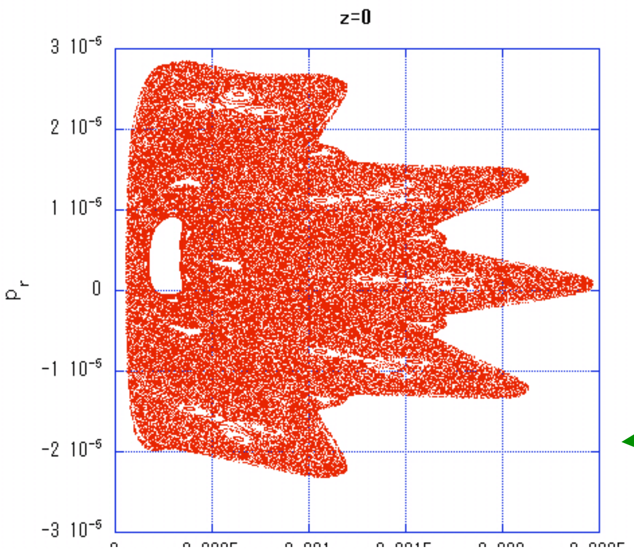
- p_θ is not constant. Variation of p_θ for small tune difference, ν_x=0.285 & ν_y=0.295.

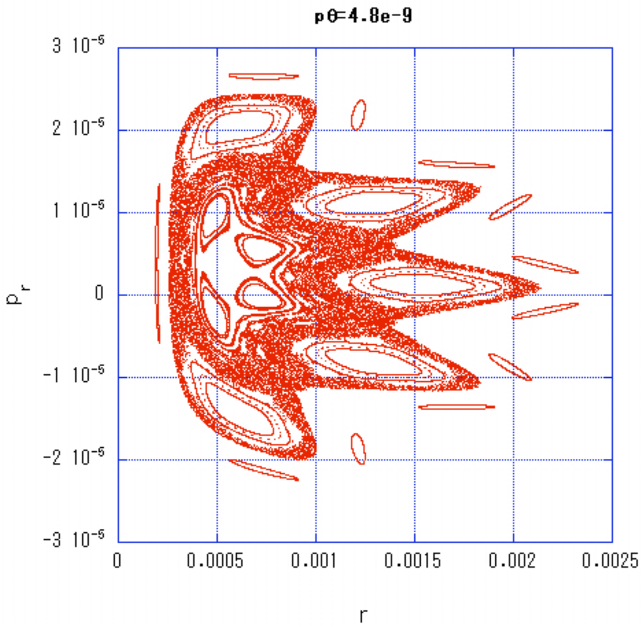


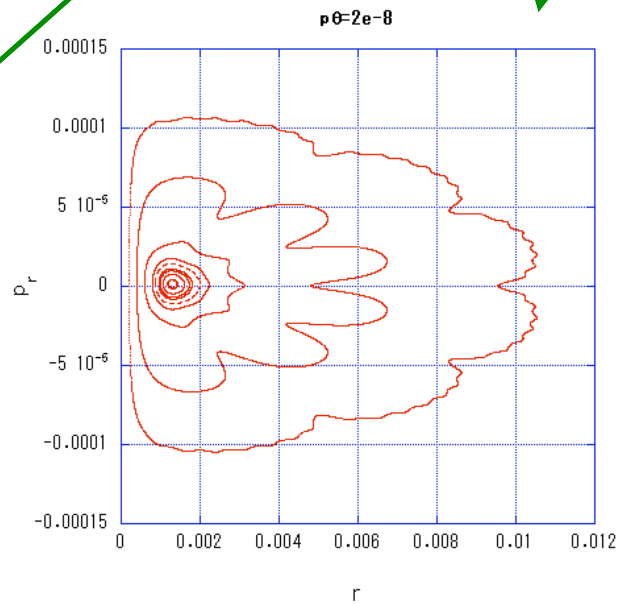


KAM for various p_θ for equal tune,.

$p_\theta = 1.2 \times 10^{-9}, 4.8 \times 10^{-9}, 2 \times 10^{-8}$.



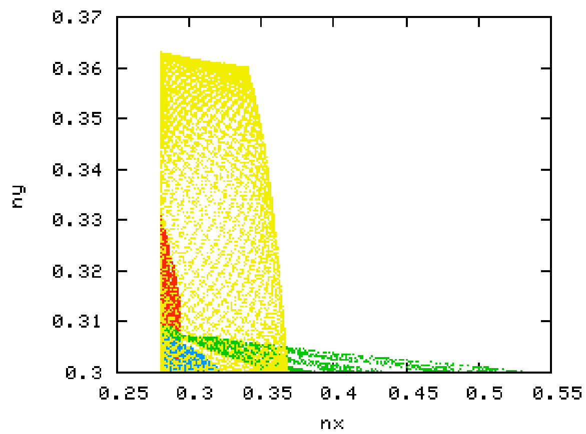


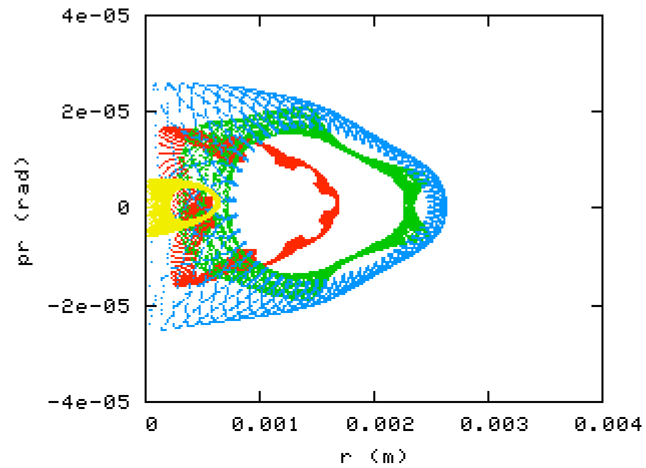


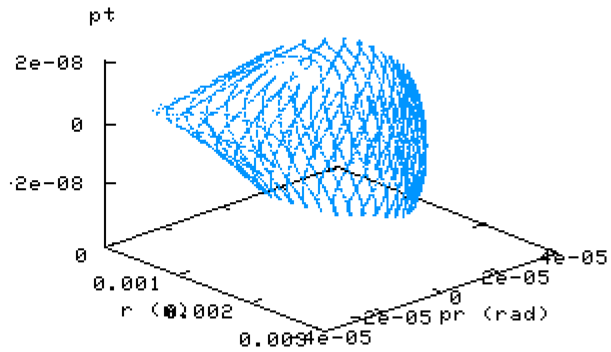
Note σ_r=0.89 mm

Motion in the phase space and tune space – example I

- Near integrable trajectory---nondiffusive

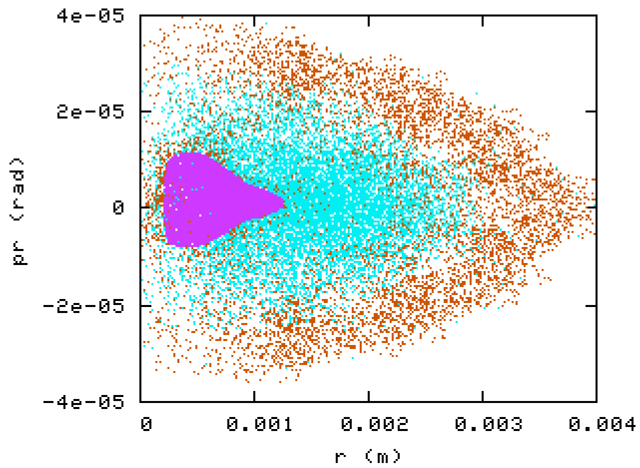


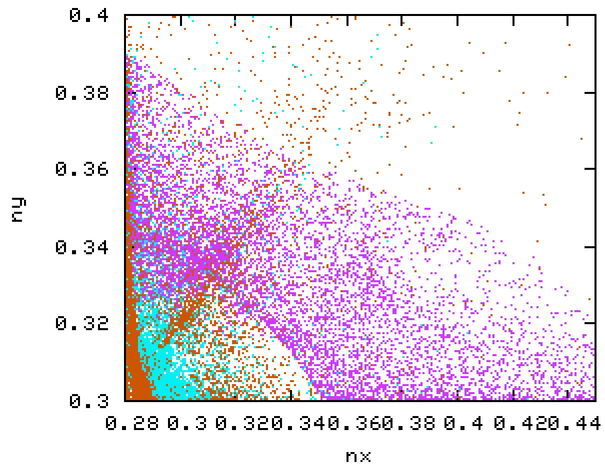


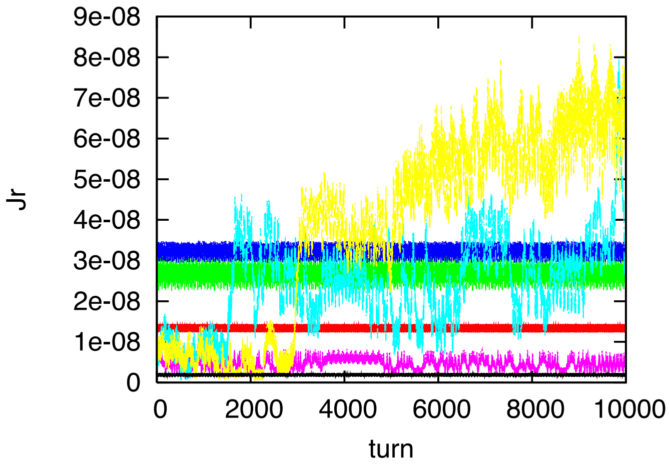


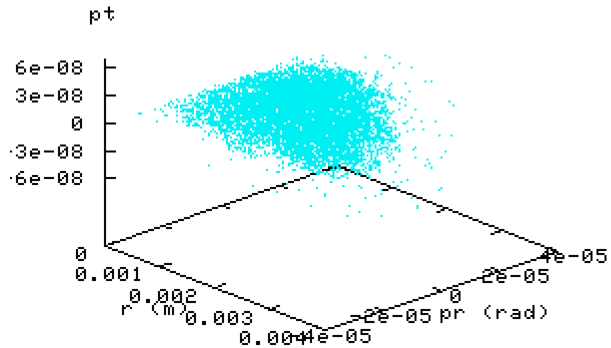
Motion in the phase space and tune space - example II

- Chaotic trajectory --- diffusive





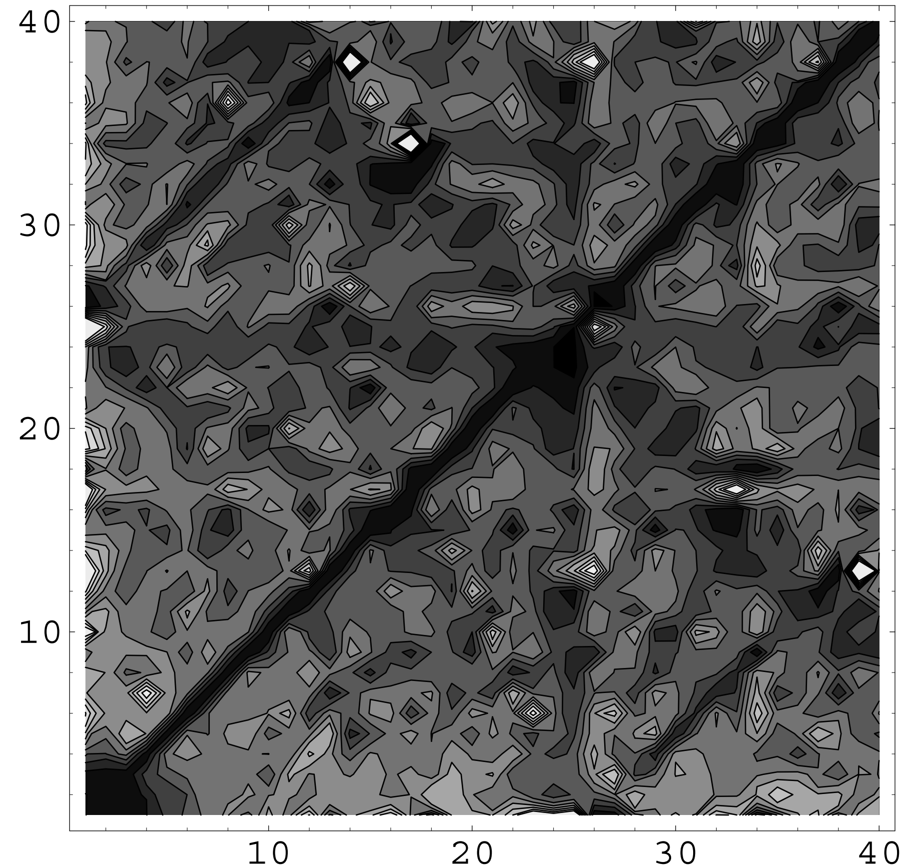




Tune scan for 3 degrees of freedom

- Tune scan without synchrotron motion

0.45, 0.45



0.05, 0.05

- Pumping machanism
- Resonance
- Separatrix crossing ($\delta\nu \sim 0$)

More

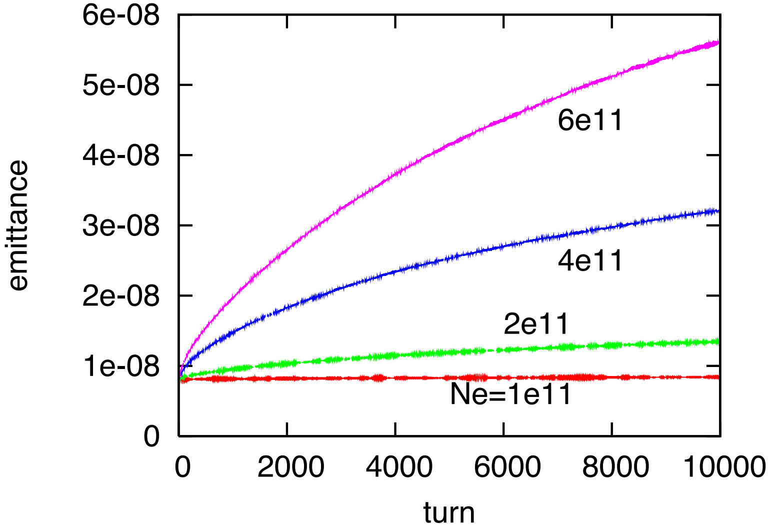
- 4 degrees of freedom – actual weak-strong model.

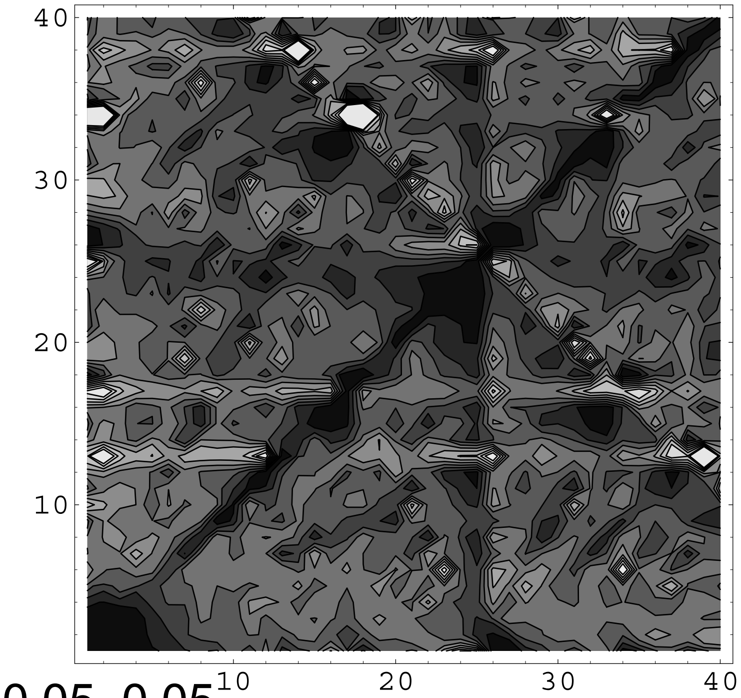
$$H p p \frac{\mu\mu\alpha\mu}{222} \left(\right)_{rz} \frac{p_\theta^2}{r^2} \frac{r r^{22}}{\beta\beta\alpha} \cos 2()(,) \left(\right) \left(\right)^2$$

- Colliding beam – $4 \times 2 \times N_{+-}$ degrees of freedom
- 3 degree of freedom+synchrotron motion. Modulation diffusion, stochastic pumping with separatrix crossing.

4 degrees of freedom

- Tune scan with synchrotron motion, ν_z =0.006.
- Vertical emittance growth.
- Resonance, m ν_y=n is seen.
- Emittance growth is large at cross points of resonances





Importance of Lattice

- Nonlinearity of beam-cloud interaction
- Integrated the nonlinear terms with multiplying β function and cos (sin) of phase difference

$$M = e^{-:U_1:} e^{-:F_{12}:} e^{-:U_2:} e^{-:F_{23}:} e^{-:U_3:} e^{-:F_{34}:} e^{-:U_4:} e^{-:F_{45}:} \ldots e^{-:F_{n1}:}$$

$$\approx e^{-:F_{11}:} \exp\left(:\sum_{i=1}^{n} U\left(e^{-:F_{1i}:} M_T \mathbf{x}\right):\right)$$

$$k_m x^m \Rightarrow k_m J^{m/2} \beta_i^{m/2} \cos(m \Delta\psi_{i1})$$

F: lattice transformation

Nonlinear term should be evaluated with considering the beta function and phase of position where electron cloud exists.

Unphysical cancel of nonlinear term may be caused by simple increase of interaction point.

Beam-beam limit

- 4 degree of freedom
- Interaction during collision. If $\sigma_z \sim \beta_y$, $\Delta\phi_\beta$ =1 rad, 4^{th}-order term $4\Delta\phi_\beta$.

$$U(x,y,z)=-\frac{r_e}{\gamma}\int_0^\infty \frac{\exp\left(-\frac{x^2}{2\sigma_x^2+u}-\frac{y^2}{2\sigma_y^2+u}\right)}{\sqrt{(2\sigma_x^2+u)(2\sigma_y^2+u)}}du$$

$$e^{-:F_1:}=\exp\left(:\sum_{i=1}^{n} U_i(M_i^T \mathbf{x}):\right)$$

$$H=\mu_x J_x+\mu_y J_y+\mu_z J_z+U_T(x,y,z,\delta)$$

Integrability near half integer tune

Reduction of the degree of freedom.

- For νx~0.5, x-motion is integrable.

(work with E. Perevedentsev)

$$\lim_{\nu_x \to 0.5} D_{Cy} = 0$$

if zero-crossing angle and no error.

$$L \propto \frac{1}{\Delta\nu_x \quad \text{(crossing angle)+(coupling)+(fast noise)}}$$

- Dynamic beta, and emittance

$$\lim_{\nu_x \to 0.5} \left\langle x^2 \right\rangle < \sigma_{x,0}^2 \qquad \lim_{\nu_x \to 0.5} \left\langle p_x^2 \right\rangle = \infty$$

- Choice of optimum ν_x

X-px plot near half integer in x

- ν_x=0.503, 0.510, 0.520, 0.540

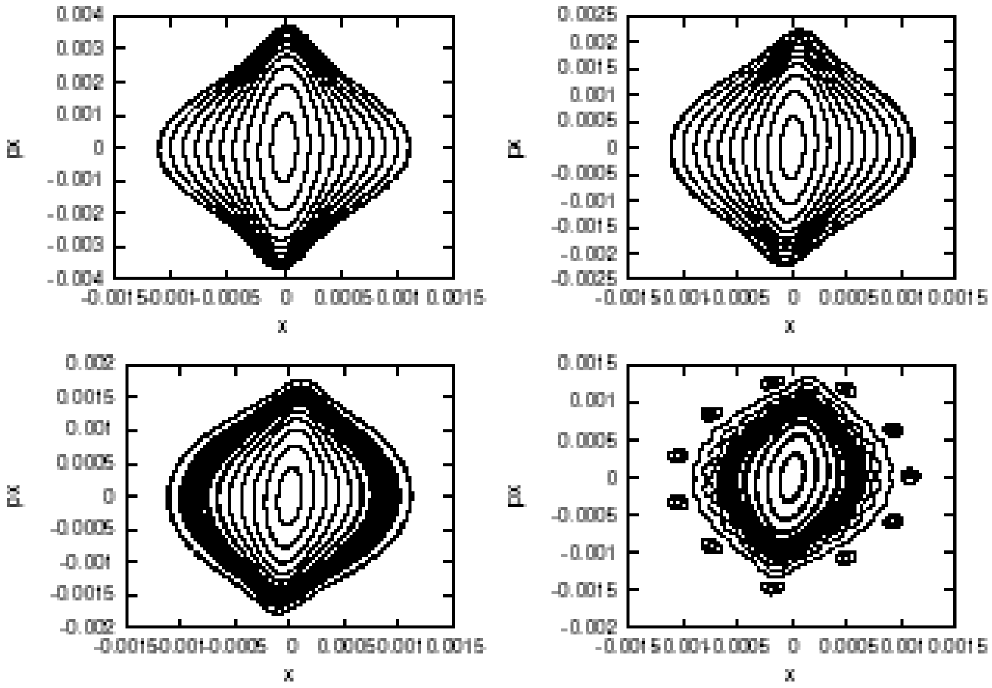


Figure 3: Phase space plot in $x - p_x$. $y_0 = 2\mu$ m$\approx 3\sigma_y$. plots (a), (b), (c) and (d) is given for ν_x =0.503,0.51,0.52 and 0.54, respectively.

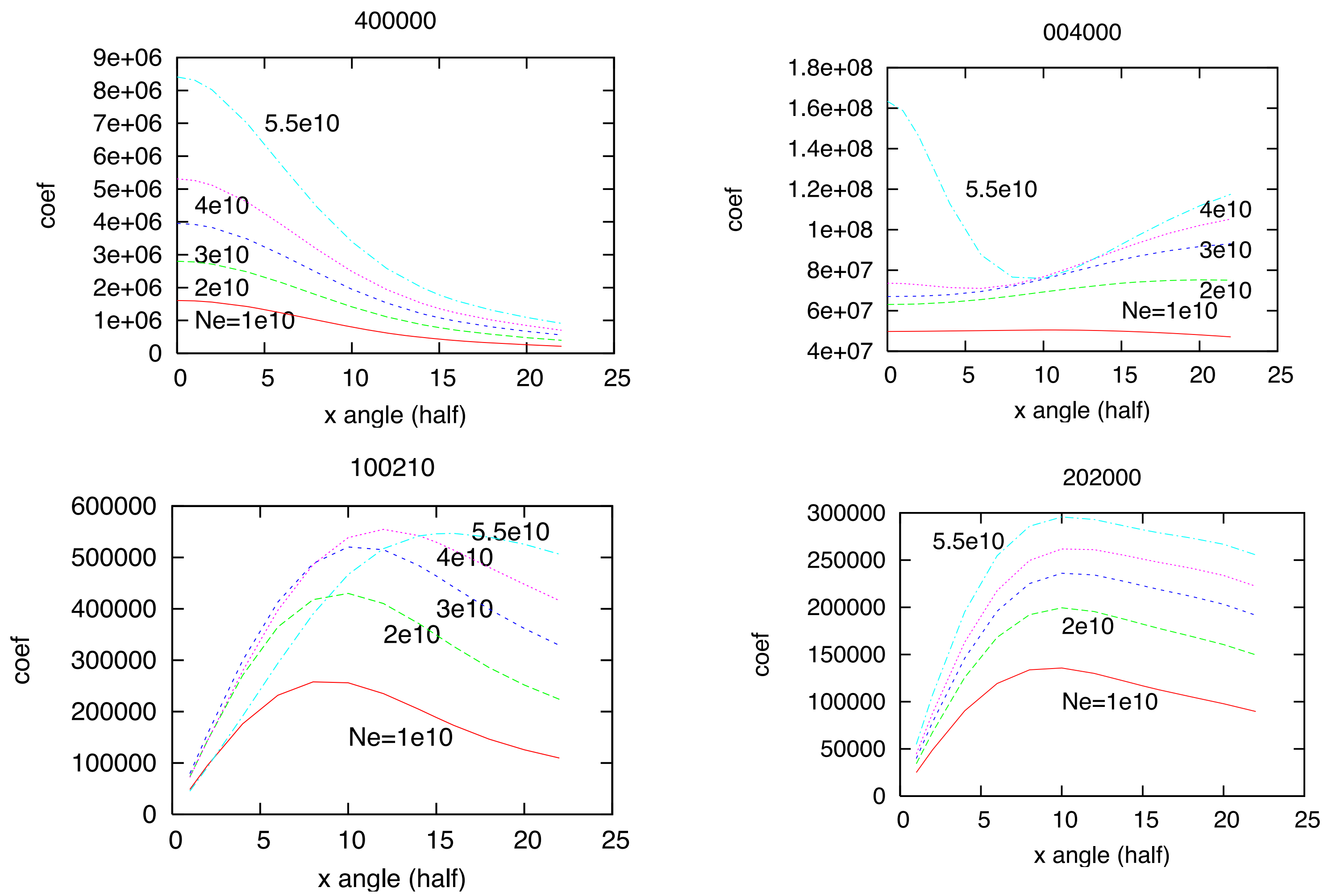
Crossing angle

- Calculate U_T using Taylor map (Diff. algebra)
- Taylor coefficient ~ Fourier coefficient, U_{klm}.

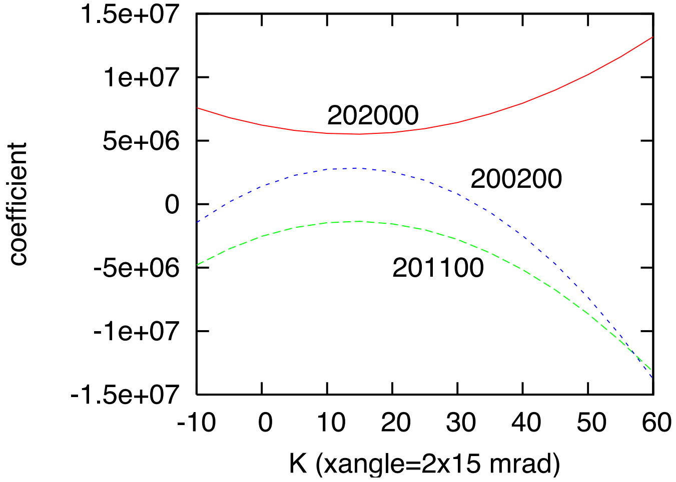
$$U_T = \sum_{klm=0}^{\infty} U_{klm} \cos(k\psi_x + l\psi_y + m\psi_z)$$

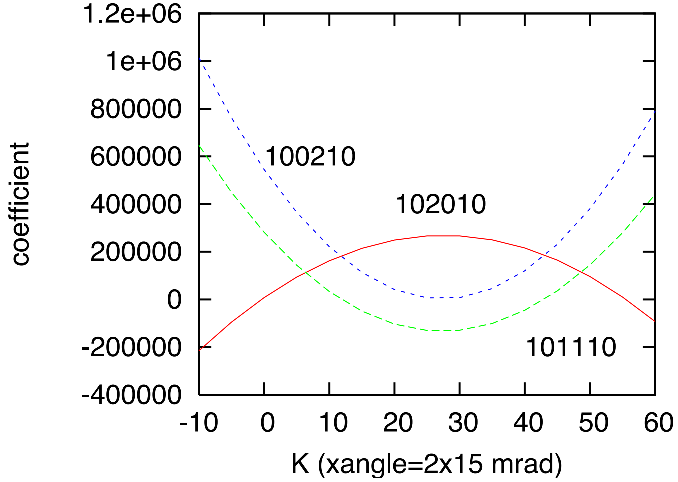
4-th order Coefficients due to crossing angle

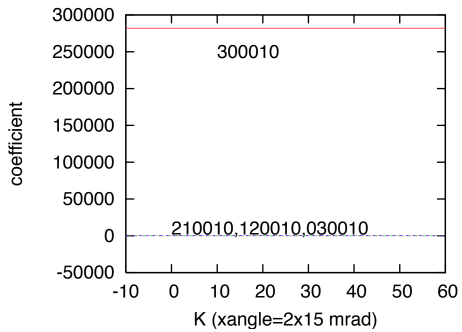
- Short bunch σ_z=3mm, $\sigma_x/\phi\sigma_z$=1 at 2x15 mrad, original super KEKB.

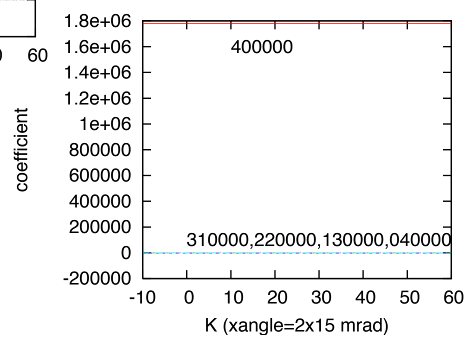


4-th order Coefficients as a function of crab sextupole strength, short bunch σ_z=3mm, $\sigma_x/\phi\sigma_z$=1









- $H=K \times p_y^2/2$, theoretical optimum, K=1/xangle.
- Clear structure- 220,121
- Flat for sextupole strength- 400, 301, 040

Crab crossing and crab waist

- Crossing angle induces synchro-beta and odd coupling resonance terms.
- Merit of Crab crossing is the absence of the terms.
- Crab waist reduces the odd coupling resonance term, but keeps the synchro-beta term.
- In the both method, Luminosity performance is improved.

Space charge limit

- Similar as electron cloud, integrate the nonlinear interaction along the ring.
- U_T: Gaussian?

$$e^{:-F_1:} = \exp(:\ :) = \exp\left(:\sum_{i=1}^{n} U_{iT}\, e^{M^{-1}} :\right) \mathbf{x}$$

$$H = \mu_x J_x + \mu_y J_y + \mu_z J_z + \delta_P(s)\, U_T(x, y, \delta, z)$$

Summary

Keyword of SAD

Dynamic aperture

Emittance