Parametrization of X-Y coupling

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Overview

Introduction

Based on discussions in J-PARC

- J-PARC operated at $(nu_x, \nu_y) = (22.40, 20.75)$ Sum resonance $\nu_x + \nu_y =$. is source of beam loss.
- New operating point (21.4,21.3) (closed to differential resonance) is tried to boost higher beam power.
- Resonance behavior for space charge tune spread.
- Tune scan measurement has been performed in SuperKEKB.
- Essential is treatment of 4×4 transfer/revolution matrix.

4×4 revolution matrix

Skew components are contained in a ring.

$$M(s) = M_{2\times 2}(s_0, s_{N_{SQ}}) \prod_{i=}^{N_{SQ}} M_{SQ}(s_i) M_{2\times 2}(s_i, s_{i-1}) \quad (1)$$

M_{SQ} is skew quadrupole component with thin lens approximation. The skew component is assumed weak.

$$M_{SQ}(s_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_i & 0 \\ 0 & 0 & 1 & 0 \\ k_i & 0 & 0 & 1 \end{pmatrix} \quad (2)$$

Two parametrizations

- Skew parametrization (perturbative)

$$M(s_0) = M_{2\times2}(s_0) M_{SQ}(s_0) \tag{3}$$

$$M_{SQ}(s_0) = \prod_{i=1}^{N_{SQ}} M_{2\times2}^{-1}(s_i, s_0) M_{SQ}(s_i) M_{2\times2}(s_i, s_0)$$

- R matrix parametrization (normal form).

$$M(s_0) = R(s_0) M_{2\times2}(s_0) R^{-1}(s_0) \tag{4}$$

$M_{2\times2}(s_0)$'s of two parametrization is nearly equal, but not equivalent (see later).

Normalization of β

$$U(s_0) = B^{-1}(s_0)M(s_0)B(s_0) \qquad U_{2\times 2}(s_0) = B^{-1}(s_0)M_{2\times 2}(s_0)B(s_0) \tag{5}$$

$$B = \begin{pmatrix} B_x & 0 \\ 0 & B_y \end{pmatrix} \qquad B_i = \begin{pmatrix} \frac{1}{\sqrt{\beta_i}} & 0 \\ \frac{\alpha_x}{\sqrt{\beta_i}} & \sqrt{\beta_i} \end{pmatrix} \tag{6}$$

$$U_{2\times 2} = \begin{pmatrix} U_x & 0 \\ 0 & U_y \end{pmatrix} \qquad U_i = \begin{pmatrix} \cos\mu_i & \sin\mu_i \\ -\sin\mu_i & \cos\mu_i \end{pmatrix} \tag{7}$$

where $\mu_i = 2\pi\nu_i$, $i = x, y$.

$$U_{SQ}(s_0) = B^{-1}(s_0)M_{SQ}(s_0)B(s_0) \qquad R_B(s_0) = B^{-1}(s_0)R(s_0)B(s_0) \tag{8}$$

Skew parametrization

$$U_{SQ}(s_0) = B^{-1}(s_0)M_{SQ}(s_0)B(s_0) = \begin{pmatrix} p_0 I_2 & -S_2 P_{SQ}^t S_2 \\ -P_{SQ} & p_0 I_2 \end{pmatrix} \tag{9}$$

$$P_{SQ} = \begin{pmatrix} -S_+ + S_- & C_+ - C_- \\ C_+ + C_- & S_+ + S_- \end{pmatrix} \tag{10}$$

$p_0 = \sqrt{1 - |P_{SQ}|}$ for symplecticity requirement.

$$S_\pm = \frac{1}{2}\int_s^{s+L} k_i \sqrt{\beta_x \beta_y} \sin\phi_\pm ds \tag{11}$$

$$C_\pm = \frac{1}{2}\int_s^{s+L} k_i \sqrt{\beta_x \beta_y} \cos\phi_\pm ds \tag{12}$$

where $\phi_\pm = \phi_x \pm \phi_y$.

Exercise: check these formulae.

R matrix formalism (beta normalization)

$$R = \begin{pmatrix} r_0 & 0 & r_4 & -r_2 \\ 0 & r_0 & -r_3 & r_1 \\ -r_1 & -r_2 & r_0 & 0 \\ -r_3 & -r_4 & 0 & r_0 \end{pmatrix} \qquad R_B = \begin{pmatrix} r_0 & 0 & q_4 & -q_2 \\ 0 & r_0 & -q_3 & q_1 \\ -q_1 & -q_2 & r_0 & 0 \\ -q_3 & -q_4 & 0 & r_0 \end{pmatrix} \tag{13}$$

where $r_0 = \sqrt{1 - r_1 r_4 + r_2 r_3}$. Relation of r_i and q_i

$$q_1 = \frac{r_1\beta_x - r_2\alpha_x}{\sqrt{\beta_x\beta_y}} \qquad q_4 = \frac{r_4\beta_y + r_2\alpha_y}{\sqrt{\beta_x\beta_y}} \tag{14}$$

$$q_2 = \frac{r_2}{\sqrt{\beta_x\beta_y}} \qquad q_3 = \frac{r_3\beta_x\beta_y + r_1\beta_x\alpha_y - r_4\alpha_x\beta_y - r_2\alpha_x\alpha_y}{\sqrt{\beta_x\beta_y}} \tag{15}$$

Relation of P_{SQ} and R_B

From

$$U_{2\times 2}(s_0)U_{SQ}(s_0) = R_B(s_0)U_{2\times 2}(s_0)R_B^{-1}(s_0), \tag{16}$$

thre relation of $S_\pm$, $C_\pm$ and q_i is given by

$$\begin{aligned}
S_+ &= \frac{1}{2}\left[-q_1 + q_4 - r_0(q_2+q_3)\sin\mu_+ + r_0(q_1-q_4)\cos\mu_+\right] \\
C_+ &= \frac{1}{2}\left[q_2 + q_3 - r_0(q_1-q_4)\sin\mu_+ - r_0(q_2+q_3)\cos\mu_+\right] \\
S_- &= \frac{1}{2}\left[q_1 + q_4 + r_0(q_2-q_3)\sin\mu_- - r_0(q_1+q_4)\cos\mu_-\right] \\
C_- &= \frac{1}{2}\left[-q_2 + q_3 + r_0(q_1+q_4)\sin\mu_- + r_0(q_2-q_3)\cos\mu_-\right]
\end{aligned} \tag{17}$$

where $\mu_\pm = \mu_x \pm \mu_y$

Behavior near sum resonance

For simplicity, consider $S_+ = C_+ = 0$, $\mu_x = \mu + \epsilon$, $\mu_y = -\mu + \epsilon$

$$U = \begin{pmatrix} U(\mu+\epsilon) & 0 \\ 0 & U(-\mu+\epsilon) \end{pmatrix} \begin{pmatrix} p_0 & 0 & -S_- & -C_- \\ 0 & p_0 & C_- & -S_- \\ S_- & -C_- & p_0 & 0 \\ C_- & S_- & 0 & p_0 \end{pmatrix} \tag{18}$$

where $p_0 = \sqrt{1 + S_-^2 + C_-^2}$.
Eigenvalues

$$\lambda_\pm = \frac{\cos\mu \pm i\sqrt{S_-^2 + C_-^2 + \sin^2\mu}}{e^{\pm(\mp)i\epsilon}\sqrt{1 + S_-^2 + C_-^2}} \tag{19}$$

Stable because of $|\lambda_\pm| = 1$

Behavior near sum resonance ($S_- = C_- = 0$)

$$U = \begin{pmatrix} U(\mu+\epsilon) & 0 \\ 0 & U(-\mu+\epsilon) \end{pmatrix} \begin{pmatrix} p_0 & 0 & -S_+ & C_+ \\ 0 & p_0 & C_+ & S_+ \\ -S_+ & C_+ & p_0 & 0 \\ C_+ & S_+ & 0 & p_0 \end{pmatrix} \qquad (20)$$

Eigenvalues

$$\lambda_\pm = e^{\pm i\mu} \frac{\cos\epsilon \pm i\sqrt{\sin^2\epsilon - S_+^2 - C_+^2}}{\sqrt{1 - S_+^2 - C_+^2}} \to e^{\pm i\mu} \frac{1 \pm \sqrt{S_+^2 + C_+^2}}{\sqrt{1 - S_+^2 - C_+^2}} \qquad (21)$$

Unstable because of $|\lambda_\pm| \neq 1$.

Near sum resonance, S_+, C_+ induce instability.

Behavior near differential resonance

For $S_- = C_- = 0$

$$\lambda_\pm = \frac{\cos\mu \pm i\sqrt{\sin^2\mu + S_+^2 + C_+^2}}{e^{\pm\epsilon}\sqrt{1 + S_+^2 + C_+^2}} \tag{22}$$

Stable because of $|\lambda_\pm| = 1$
For $S_+ = C_+ = 0$

$$\lambda_\pm = e^{i\mu}\frac{\cos\epsilon \pm i\sqrt{S_-^2 + C_-^2 + \sin^2\epsilon}}{\sqrt{1 + S_-^2 + C_-^2}} \tag{23}$$

Stable because of $|\lambda_\pm| = 1$. Two tunes split $\tan\delta\mu/2 = \sqrt{S_-^2 + C_-^2}$.
Near diff resonance, S_-, C_- induce tune split.

Emittance and Tune of Hor. and Ver. mode

1. Betatron oscillation is characterized by 4×4 revolution/transver matrix.
2. The revolution matrix has 4 eigenvalues, $\exp(\pm i\mu_X)$ and $\exp(\pm i\mu_Y)$, where $\mu_i = 2\pi\nu_i$.
3. We measure the two tune, (ν_X, ν_Y)
4. Meaningless variables, $\nu_x, \nu_y, \varepsilon_x, \varepsilon_y, \beta_x, \beta_y, \sigma_X, \sigma_Y$

Emittance

Beam envelope normalized by β, $B^{-1}\langle \boldsymbol{x}\boldsymbol{x}^T\rangle B$, is regarded as the ordinary emittance, if it is diagonalized by $(\varepsilon_x, \varepsilon_x, \varepsilon_y, \varepsilon_y)$.

$$
\begin{aligned}
B^{-1}\langle \boldsymbol{x}\boldsymbol{x}^T\rangle B &= \varepsilon_X \begin{pmatrix} r_0^2 & & & \\ 0 & r_0^2 & & \\ -r_0 q_1 & -r_0 q_2 & q_1^2+q_2^2 & \\ -r_0 q_3 & -r_0 q_4 & q_1 q_3 + q_2 q_4 & q_3^2+q_4^2 \end{pmatrix} \\
&+ \varepsilon_Y \begin{pmatrix} q_4^2 q_2^2 & & & \\ q_3 q_4 + q_1 q_2 & q_3^2+q_1^2 & & \\ r_0 q_4 & -r_0 q_3 & r_0^2 & \\ -r_0 q_2 & q_1 & 0 & r_0^2 \end{pmatrix}
\end{aligned} \tag{24}
$$

For $\varepsilon_Y = 0$, $\varepsilon_y = \varepsilon_X(q_1^2+q_2^2)$, $\varepsilon_X(q_3^2+q_4^2)$ or determinant of (3,4) submatrix $\varepsilon_X\sqrt{(q_1^2+q_2^2)(q_3^2+q_4^2)-(q_1q_3+q_2q_4)^2}$. that is, it is not unique, nor is not invariant for s, exactly speaking.

Behavior of the coupling parameter q_i for s

Region without skew components

$$R_B(s_2) = U(s_2, s_1) R_B(s_1) U(s_2, s_1)^{-1} \tag{25}$$

$$U(s_2, s_1) = \begin{pmatrix} U_x & 0 \\ 0 & U_y \end{pmatrix} \qquad U_i = \begin{pmatrix} \cos\phi_i(s_2, s_1) & \sin\phi_i(s_2, s_1) \\ -\sin\phi_i(s_2, s_1) & \cos\phi_i(s_2, s_1) \end{pmatrix} \tag{26}$$

where $\phi_i(s_2, s_1)$ is the betatron phase difference.

$$\begin{aligned} q_1(s_2) + q_4(s_2) &= (q_1 + q_4)\cos\phi_- + (q_2 - q_3)\sin\phi_- \\ q_2(s_2) - q_3(s_2) &= (q_2 - q_3)\cos\phi_- - (q_1 + q_4)\sin\phi_- \\ q_2(s_2) + q_3(s_2) &= (q_2 + q_3)\cos\phi_+ - (q_1 - q_4)\sin\phi_+ \\ q_1(s_2) - q_4(s_2) &= (q_1 - q_4)\cos\phi_+ + (q_2 + q_3)\sin\phi_+ \end{aligned} \tag{27}$$

where $\phi_\pm = \phi_x \pm \phi_y$

Invariant of the coupling parameters

Region without skew components

$$(q_1 + q_4)^2 + (q_2 - q_3)^2 \qquad (q_1 - q_4)^2 + (q_2 + q_3)^2 \tag{28}$$

or

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 \qquad q_1 q_4 - q_2 q_3 \tag{29}$$

Tune scan experiment

Tune split

$$\delta\mu = 2\sqrt{S_-^2 + C_-^2} \tag{30}$$

Using Eq.(17).

$$\begin{aligned}\delta\mu &= [(q_1+q_4)(1-\cos\mu_-) + (q_2-q_3)\sin\mu_-]^2 \\ &\quad + [-(q_2-q_3)(1-\cos\mu) + [(q_1+q_4)\sin\mu]^2 \\ &= 4\left[(q_1+q_4)^2 + (q_2-q_3)^2\right]\sin^2\frac{\mu_-}{2}\end{aligned} \tag{31}$$

Vertical emittance induced by x-y coupling in a very rough expression

$$\frac{\varepsilon_y}{\varepsilon_x} \approx \frac{q_1^2+q_2^2+q_3^2+q_4^2}{2} \approx \frac{\delta\mu}{8\sin^2\frac{\mu_-}{2}} \tag{32}$$

Coupling resonance correction using Skew Quads

Coupling resonance correction, not coupling correction.

Several skew quads are added. The whole transformation in revolution matrix is expressed by N_{SQ}, V_{SQ}.

$$M_0(s)N_{SQ} = BUU_{SQ}V_{SQ}B^{-1} \qquad V_{SQ} = BN_{SQ}B^{-1} \tag{33}$$

$U_{SQ}V_{SQ}$ is parametrized by

$$U_{SQ}V_{SQ} = \begin{pmatrix} p'_0 I_2 & -S_2 P'^t_{SQ} S_2 \\ -P'_{SQ} & p'_0 I_2 \end{pmatrix} \tag{34}$$

We assume N_{SQ}, V_{SQ} as an example,

$$V_{SQ} = \begin{pmatrix} 1 & 0 & -K_2 & 0 \\ 0 & 1 & K_1 & 0 \\ 0 & 0 & 1 & 0 \\ K_1 & K_2 & K_1 K_2 & 1 \end{pmatrix} \tag{35}$$

$K_i = \sqrt{\beta_{x,i}\beta_{y,i}}k_i$. Two skews are installed at $\Delta\phi_x = 0.25, \Delta\phi_y = 1$.

Skew Quads strength for the resonance correction

$U_{SQ}V_{SQ}$ gives

$$P'_{SQ} = \begin{pmatrix} -S_+ - S_- - K_2 & C_+ - C_- \\ C_+ + C_- + K_1 & S_+ - S_- \end{pmatrix} \tag{36}$$

Differential resonance correction is performed by $K_1 = -2C_-, K_2 = -2S_-$, while some resonance terms become $C'_+ = C_+ - C_-, S'_+ = S_+ - S_-$.
Some resonance correction is performed by $K_1 = -2C_+, K_2 = -2S_+$.
V_{SQ} using 4 skew quads can compensate both of sum and differential resonances, corresponding $|R(s_1)| = 0$

$$V_{SQ} \approx \begin{pmatrix} 1 & 0 & -K_2 & -K_4 \\ 0 & 1 & K_1 & K_3 \\ -K_3 & -K_4 & 1 & 0 \\ K_1 & K_2 & 0 & 1 \end{pmatrix} \tag{37}$$

4 skew quads are located at $\Delta\phi_{x,21} = 0.25, \Delta\phi_{y,21} = 1$,
$\Delta\phi_{x,31} = 1, \Delta\phi_{y,31} = 0.25$ $\Delta\phi_{x,41} = 0.25, \Delta\phi_{y,41} = 0.25$.

Vertical mode emittance, ε_Y

Vertical mode emittance ε_Y is induced by dispersion η_Y, η'_Y. Measureable dispersions are $(\eta_x, \eta'_x, \eta_y, \eta'_y)$.

$$\mathcal{H}_Y = \gamma_Y \eta_Y^2 + 2\alpha_Y \eta_Y \eta'_Y + \beta_Y \eta'^2_Y \tag{38}$$

Summary

- Relation of Skew and R matrix parametrizations is obtained.
- Vertical emittance (ε_y) from leak of horizontal mode emittance (ε_X) is expressed by R matrix elements.
- The vertical emittance (ε_y) is estimated by tune stopband measurement.
- R parameters at several locations (IP...) are controlled by Skew Quads.
- Coupling resonance compensation is performed by Skew Quads in the same meaning.
- Vertical mode emittance (ε_Y) is induced by dispersion $\eta_Y(\eta_x, \eta_y, R)$.

The End

Thank you for your attention